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# Functional determinants, generalized BTZ geometries and Selberg zeta function

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## Abstract

We continue the study of a special entry in the AdS/CFT dictionary, namely a holographic formula relating the functional determinant of the scattering operator in an asymptotically locally anti-de Sitter space to a relative functional determinant of the scalar Laplacian in the bulk. A heuristic derivation of the formula involves a one-loop quantum effect in the bulk and the corresponding sub-leading correction at large  $N$  on the boundary. We presently explore the formula in the background of a higher dimensional version of the Euclidean BTZ black hole, obtained as a quotient of hyperbolic space by a discrete subgroup of isometries generated by a loxodromic (or hyperbolic) element consisting of dilation (temperature) and torsion angles (twist). The bulk computation is done using heat-kernel techniques and fractional calculus. At the boundary, we acquire a recursive scheme that allows us to successively include rotation blocks in spacelike planes in the embedding space. The determinants are compactly expressed in terms of an associated (Patterson–) Selberg zeta function and a connection to quasi-normal frequencies is discussed.

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## 1. Introduction

The AdS/CFT correspondence has been a very productive area of research ever since its appearance, more than a decade ago, in the form of Maldacena's conjecture together with a calculational prescription [1–3]. Many developments are, perhaps more appropriately, embraced under the name *gauge/gravity duality* for they depart from the original canonical examples that involved anti-de Sitter spacetime and the bulk side is usually restricted to the (super)gravity approximation. On the one hand, these canonical (or most symmetric) cases seem to be grounded in solid mathematical foundations, such as harmonic analysis on symmetric spaces [4], representation theory [5], and hyperbolic and conformal geometry [6].

On the other hand, recent and exciting applications to physically relevant situations (see e.g. [7, 8]) are more heuristic, and therefore mathematically exact results become rare. This starts already when including finite temperature on the boundary theory, leading to AdS black holes as bulk background geometry.

One notable exception is the BTZ black hole [9, 10]. While having essentially the same standard features of higher dimensional AdS black holes, its virtue of being a space of constant negative curvature greatly simplifies many derivations (cf [11]). Here one can include temperature and still obtain explicit analytic results for the Green's functions, spectrum of quasi-normal modes etc, and several holographic features have a mathematical counterpart [12, 13].

Another instance that transcends its original frame is the case of a holographic formula relating the functional determinant of the scattering operator in an asymptotically locally anti-de Sitter (ALAdS) space to a relative functional determinant of the scalar Laplacian in the bulk. The conformal anomaly at the boundary at leading large  $N$  can be read from the regularization of the classical gravitational action; a quantum correction to this classical gravitational action such as the one-loop contribution of a scalar field corresponds to a sub-leading effect at the boundary. This was confirmed in [14, 15]. However, the mapping can be further extended to an equality between functional determinants [16, 17], namely

$$\frac{\det_{-}(\Delta_X - \lambda(n - \lambda))}{\det_{+}(\Delta_X - \lambda(n - \lambda))} = \det S_M(\lambda), \quad (1)$$

where  $\Delta_X$  is the Laplacian operator in the bulk and  $S_M(\lambda)$  stands for the Euclidean two-point correlation function of the boundary operator  $O_\lambda$  dual to the bulk scalar field  $\Phi$ . The label '+' refers to the usual determinant, obtained for example via heat kernel techniques, whereas the label '-' refers to the analytic continuation from  $\lambda$  to  $n - \lambda$ .

In this work, we bring together these two instances to explore the holographic formula in a generalized BTZ geometry. This has been previously done for thermal AdS and for a related BTZ geometry [18], but we realize that the bulk computation can be readily extended to the more general case covered by a result due to Patterson [19] in terms of an associated Selberg zeta function for the resulting BTZ geometry that combines dilation with twist in an internal sphere. The challenge we presently undertake is to elucidate how this general result can be recovered on the boundary.

Regarding the 'separate lives' of each side of the holographic formula, a few remarks are worth mentioning. The functional determinant on the bulk side is the central object when computing one-loop effective actions<sup>1</sup>. For example, in BTZ<sub>3</sub> the effective action had been computed by Mann and Solodukhin [21], and later Perry and Williams noticed a connection with the Selberg zeta function. The functional determinant on the boundary, in turn, involves a pseudo-differential operator and has been less explored in the physics literature; however, it is related to conformal powers of the Laplacian (cf [17]) and one can find a connection with the Selberg zeta function in multi-loop amplitudes for the bosonic string [22]. AdS/CFT correspondence, via the holographic formula in BTZ<sub>3</sub>, connects these two *a priori* unrelated computations.

The outline of this paper is as follows. The quotient geometry and a general ansatz for the bulk metric is reviewed in the next section. The computation of the determinant on the rhs of the holographic formula, using a particular parametrization of the compact boundary, is performed in section 3. For completeness, a new derivation of the result by Patterson

<sup>1</sup> As mentioned, one-loop corrections correspond to sub-leading terms in the large- $N$  expansion on the boundary and any refinement of the holographic correspondence to include them leads to such bulk determinants. This opens the possibility of finding interesting effects in strongly coupled field theories, not captured by the leading order, as in [20].

for the lhs, using this time heat-kernel and fractional calculus, is included in section 4. As an application of the general results of the previous sections, the holographic nature of the determinant of the Laplacian on the torus is illustrated in section 5. Section 6 elaborates on properties of the Selberg zeta function to elucidate a connection with quasi-normal frequencies and with Barnes' multiple zeta and gamma functions. Section 7 includes concluding remarks and perspectives. Four appendices provide some supplementary material.

## 2. Quotient geometry

The existence of the BTZ solution in the form of a quotient in three dimensions steered out the search for higher dimensional analogs [23–26]. Indeed, one can foresee, by the rich geometrical structure of the negative curvature manifolds such as  $AdS_d$  [27], the presence of interesting solutions of the form  $\Gamma \backslash AdS_d^*$ , where  $AdS_d^*$  is a section of  $AdS_d$  and  $\Gamma$  is a discrete subgroup of the AdS isometry group. A classification of the possible identifications can be found in [28, 29] and references therein. Locally, these spaces can be written as

$$ds^2 = B(r)^2(d\hat{x}^i d\hat{x}^j \hat{g}_{ij}) + C(r)^2 dr^2 + A(r)^2(d\tilde{y}^m d\tilde{y}^n \tilde{g}_{mn}), \tag{2}$$

where  $\hat{g}_{ij}$  and  $\tilde{g}_{mn}$  are intrinsic metrics of two constant curvature manifolds, hereinafter referred to as worldsheet and transverse section, respectively.

The functional determinants in this paper, however, require the Euclidean formulation. The bulk space is then locally asymptotically hyperbolic and the conformal boundary is a compact Riemannian manifold. The quotients to be considered can be described in the Poincaré half-plane,

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2}, \tag{3}$$

with the identification  $\Gamma$  generated by a hyperbolic element of the discrete group of isometries whose action is

$$(z, \vec{x}) \sim e^l(z, \mathbb{A} \vec{x}). \tag{4}$$

Here  $\mathbb{A}$  is a twist or rotation, which can be cast in block form with eigenvalues  $e^{\pm i\varphi_k}$  with  $k = 1, \dots, K \leq \lfloor n/2 \rfloor^2$  and an extra unit eigenvalue in case  $n$  is odd. In geometric terms, to cast  $\mathbb{A}$  in block form is equivalent to reduce the rotations to planes that do not intersect, and thus  $k$  also labels the different planes where the rotations occur. If  $\mathbb{A}$  is trivial then the transverse section in equation (2) corresponds to a sphere. Otherwise the transverse section and worldsheet are intertwined manifolds.

In order to classify this identification [28] one can note that equation (4) is generated by the Killing vector

$$\xi_{BTZ} = l \left( z \frac{\partial}{\partial z} + x^i \frac{\partial}{\partial x^i} \right) + \sum_{k=1}^{\lfloor n/2 \rfloor} \varphi_k \left( x^{k_1} \frac{\partial}{\partial x^{k_2}} - x^{k_2} \frac{\partial}{\partial x^{k_1}} \right), \tag{5}$$

where the last part of the vector stands for the sum of the generators of a rotation in each of the  $(x^{k_1}, x^{k_2})$  planes.

There are several known examples. For instance one can quote [30] whose line element for the Euclidean section is given by

$$ds^2 = N(r)^2 d\Sigma_3 + N(r)^{-2} dr^2 + r^2 d\phi^2, \tag{6}$$

<sup>2</sup>  $\lfloor \alpha \rfloor$  stands for the largest integer smaller than or equal to  $\alpha$ .

with  $N^2(r) = r^2 - r_+^2$  and

$$d\Sigma_3 = \cos^2 \theta d\tau^2 + \frac{1}{r_+^2} (d\theta^2 + \sin^2 \theta d\chi^2). \quad (7)$$

The horizon in these coordinates is located at  $r = r_+$ . In this case the identification is performed along the Killing vector (B.3):

$$\xi = 2\pi r_+ \frac{\partial}{\partial \phi} = 2\pi r_+ \left( w \frac{\partial}{\partial p} + p \frac{\partial}{\partial w} \right) = 2\pi r_+ \left( z \frac{\partial}{\partial z} + y^i \frac{\partial}{\partial y^i} \right), \quad (8)$$

where the coordinates  $w$  and  $p$  are defined in appendix A. In terms of the general identification (equation (4)) this corresponds to the case  $\mathbb{A}$  trivial, i.e. no twist at all.

The continuation to Lorentzian signature in general yields regular as well as black hole spacetimes. In [28] it is argued that only the non-spinning BTZ black hole has a generalization to higher dimensions. This translates into a dilation combined with a rotation in the spatial planes in the embedding space, which restrict the entries of the matrix  $\mathbb{A}$ . These issues, though important and somewhat controversial, are not essential to our present discussion in the Euclidean context and we keep a generic rotation matrix.

### 3. Boundary

Let us start with the functional determinant on the boundary. We first need to compute the two-point correlation function of the dual operator with scaling dimension  $\lambda$ . A useful way to get this information is to examine the scattering of the bulk field in the locally asymptotically hyperbolic background.

#### 3.1. Scattering

The metric is first suitable written in the ‘scattering form’

$$g_X = dt^2 + \text{ch}^2 t du^2 + \text{sh}^2 t d\Omega_{n-1}^2, \quad (9)$$

so that the (positive) Laplacian takes the following form:

$$\Delta_X = -\frac{1}{\text{ch} t \text{sh}^{n-1} t} \partial_t (\text{ch} t \text{sh}^{n-1} t \partial_t) - \frac{1}{\text{ch}^2 t} \partial_u^2 - \frac{1}{\text{sh}^2 t} \Delta_\Omega. \quad (10)$$

One has to consider eigenfunctions of the form  $e^{i\kappa u} \times \{\text{angular part}\}$ , compatible with the identifications. For instance, when there is no mixing  $\kappa = 2\pi N/l$ ,  $N \in \mathbb{Z}$ , with only one block,  $\kappa = 2\pi N/l \pm r\varphi/l$  and so on. This leads then to an effective one-dimensional operator

$$\widehat{H}_{L,\kappa} = -\frac{1}{\text{ch} t \text{sh}^{n-1} t} d_t (\text{ch} t \text{sh}^{n-1} t d_t) + \kappa^2 \text{sech}^2 t + \gamma_L^2 \text{csch}^2 t, \quad (11)$$

where  $\gamma_L^2 = L(L + n - 2)$  are the eigenvalues of the angular part. Here one can recognize a one-dimensional stationary Schrödinger equation with a (generalized) Pöschl–Teller potential

$$H_{L,\kappa} = -d_t^2 + \alpha(\alpha + 1)\text{csch}^2 t - \beta(\beta + 1)\text{sech}^2 t \quad (12)$$

with  $\alpha = -3/2 + n/2 + L$  and  $\beta = -1/2 + i|\kappa|$ .

The scattering matrix [31] with the spectral parameter  $\lambda(n - \lambda)$  (here  $\lambda \equiv n/2 + \nu$ ) is known to be given, modulo unimportant factors, by

$$\mathbb{S}_{L,\kappa}(\lambda) = \frac{\Gamma(L/2 + n/4 + \nu/2 + i\kappa/2) \Gamma(L/2 + n/4 + \nu/2 - i\kappa/2)}{\Gamma(L/2 + n/4 - \nu/2 + i\kappa/2) \Gamma(L/2 + n/4 - \nu/2 - i\kappa/2)}. \quad (13)$$

These are the coefficients of the expansion of the scattering operator, or thermal two-point correlation function, in the chosen basis.

### 3.2. Sphere: eigenfunctions

As a technical prelude, one has to construct eigenfunctions of the Laplacian on the  $(p + q + 1)$ -sphere out of the usual harmonics for the  $p$ - and  $q$ -sphere. This is very similar to what we had before for the bulk metric in the scattering form

$$ds^2 = d\theta^2 + \cos^2 \theta d\Omega_p^2 + \sin^2 \theta d\Omega_q^2, \tag{14}$$

with the Laplacian

$$\Delta_{\Omega_{p+q+1}} = \frac{1}{\cos^p \theta \sin^q \theta} \partial_\theta (\cos^p \theta \sin^q \theta \partial_\theta) + \frac{1}{\cos^2 \theta} \Delta_{\Omega_p} + \frac{1}{\sin^2 \theta} \Delta_{\Omega_q}. \tag{15}$$

Plugging in the eigenvalues of the spherical harmonics for the little spheres (orbital quantum numbers  $r$  and  $s$ ) we end up with an effective one-dimensional equation which can be eventually related to Jacobi polynomials of orders  $m = 0, 1, 2, \dots$ . The relation with the orbital quantum number  $L$  of the  $(p + q + 1)$ -sphere is  $L = 2m + r + s$ . Here one can check that the counting of states and degeneracies agree.

### 3.3. Adding internal rotations: one block

The functional determinant we need to compute on the boundary can be cast as a trace

$$\log \det S = \text{tr} \log S = \sum_{\text{eigenstates}} \log \mathbb{S}_{L,\kappa}. \tag{16}$$

Now consider one rotation block. Following essentially the same steps as in the pure dilation case [18], i.e. taking derivative and using an integral representation of the gamma function, we write for the derivative of log-det of the scattering operator

$$\text{tr}(S^{-1} \partial_\lambda S(\lambda)) = - \sum_{\text{eigenstates}} \int_0^\infty dt \frac{e^{-tL/2 - it\kappa/2}}{1 - e^{-t}} (e^{-\lambda t/2} + e^{(\lambda-n)t/2}), \tag{17}$$

but now  $\kappa = 2\pi N/l \pm r\varphi/l$  depends not only on  $N$  but also on the orbital quantum numbers on the  $S^1$  and  $S^{n-3}$ . The  $S^{n-1}$  is decomposed into an  $S^1$ , an  $S^{n-3}$  and a polar angle  $\theta$ , so that  $L = 2m + r + s$ . The trick is to sum over  $m, r, s = 0, 1, 2, 3, \dots$ , taking into account the degeneracies

$$\sum_{N=-\infty}^\infty e^{-i\pi N t/l} \sum_{m=0}^\infty e^{-tm} \left( 1 + \sum_{r=1}^\infty e^{-rt/2} [e^{ir\varphi t/2l} + e^{-ir\varphi t/2l}] \right) \sum_{s=0}^\infty e^{-st/2} \text{deg}(n-3, s). \tag{18}$$

Now, sum up and again use Poisson summation to write in terms of deltas and take the integral, just as in [18]. The indirect contributions to the trace are then collected in the following result:

$$\text{tr}'(S^{-1} \partial_\lambda S(\lambda)) = -2l \sum_{N=1}^\infty \frac{e^{-\lambda l N} + e^{(\lambda-n)l N}}{|1 - e^{-lN+i\varphi N}|^2 \times (1 - e^{-lN})^{n-2}}. \tag{19}$$

### 3.4. Full rotation matrix $\mathbb{A}$

In case there is yet another block, one just has to replace the sum on the  $(n - 3)$ -sphere in the very same way we did to add one block. That is, in the last factor of the trace formula

(equation (18)),

$$\sum_{s=0}^{\infty} e^{-st/2} \text{deg}(n-3, s) \rightarrow \sum_{m'=0}^{\infty} e^{-tm'} \left( 1 + \sum_{r'=1}^{\infty} e^{-r't/2} [e^{ir'\phi't/2l} + e^{-ir'\phi't/2l}] \right) \times \sum_{s'=0}^{\infty} e^{-s't/2} \text{deg}(n-5, s'). \tag{20}$$

Each sphere is being decomposed into smaller ones, just as nesting dolls or *matryoshkas*. Successive applications of this procedure finally lead to the expression for  $K$  blocks:

$$\text{tr}'(S^{-1} \partial_{\lambda} S(\lambda)) = -2l \sum_{N=1}^{\infty} \frac{e^{-\lambda l N} + e^{(\lambda-n)l N}}{|1 - e^{-lN+i\phi_1 N}|^2 \dots |1 - e^{-lN+i\phi_K N}|^2 (1 - e^{-lN})^{n-2K}}, \tag{21}$$

which is precisely the bulk result, as we will see,

$$\frac{1}{2} \text{tr}'(S^{-1} \partial_{\lambda} S(\lambda)) = \frac{Z'_{\Gamma}}{Z_{\Gamma}}(\lambda) + \frac{Z'_{\Gamma}}{Z_{\Gamma}}(n - \lambda), \tag{22}$$

or, integrating,

$$\det' S(\lambda) = [Z_{\Gamma}(n - \lambda)/Z_{\Gamma}(\lambda)]^2. \tag{23}$$

#### 4. Bulk

For completeness, we examine the bulk side in order to parallel the boundary computation. The relative functional on this side of the correspondence can also be cast as a trace:

$$\log \frac{\det_{-}(\Delta_X - \lambda(n - \lambda))}{\det_{+}(\Delta_X - \lambda(n - \lambda))} = \text{tr}_{-} \log(\Delta_X - \lambda(n - \lambda)) - \text{tr}_{+} \log(\Delta_X - \lambda(n - \lambda)). \tag{24}$$

We focus on the standard +branch, take the derivative with respect to  $\lambda$  and compute in terms of the (truncated) heat kernel representation for the Green's function subtracting the direct contribution given by the Green's function for the original hyperbolic space. The trace here means taking the coincidence-point limit and integrating over the fundamental domain, so that

$$(2\lambda - n) \text{tr}(G_X^{+} - G_H^{+}) = (2\lambda - n) \int_0^{\infty} ds \text{tr} K'_X(\sigma, s) e^{s\lambda(n-\lambda)}. \tag{25}$$

The idea is to apply the method of images to compute the coincidence limit of the Green's function, that is, to sum up the contributions from image points. We start with the heat kernel for the (positive) Laplacian  $\exp(-t \Delta_H)$  in hyperbolic space  $\mathbb{H}^{N+1}$  which can be compactly written in terms of Weyl's fractional derivative [32]:

$$K_{n+1} = \frac{e^{-tn^2/4}}{(2\pi)^{n/2}} \cdot {}_x W_{\infty}^{\frac{n}{2}} [K_1]. \tag{26}$$

It is convenient to write it as a function of  $x = \text{ch } \sigma$ , where  $\sigma$  is the geodesic distance, so that  $x - 1$  is the chordal distance on the embedded hyperboloid (equation (B.1)). The input is the heat kernel on the line  $K_1 = \frac{1}{(4\pi t)^{1/2}} e^{-\sigma^2/4t}$ . Now, the geodesic distance between a point  $(z, \vec{x})$  in the fundamental region and its  $m$ th image under the identification  $(z_m, \vec{x}_m) = e^{ml} (z, \mathbb{A}^m \cdot \vec{x})$  satisfies

$$\text{ch } \sigma(z, \vec{x} \mid z_m, \vec{x}_m) = \frac{z^2(1 + e^{2ml}) + |(\mathbb{I} - e^{ml} \mathbb{A}^m) \cdot \vec{x}|^2}{2z^2 e^{ml}}. \tag{27}$$

The fundamental region can be taken as  $1 \leq z \leq e^l$  with the volume element  $d^n \vec{x} dz/z^{n+1}$  and we just need the volume integral of a function of the geodesic distance between image points in order to take the trace of the heat kernel.

We proceed in two steps. First, change variables

$$\vec{x} \rightarrow \vec{x} - \vec{x}_m \equiv \vec{y}_m = (\mathbb{I} - e^{ml} \mathbb{A}^m) \cdot \vec{x} \tag{28}$$

with Jacobian  $\frac{\partial \vec{y}_m}{\partial \vec{x}} = \mathbb{I} - e^{ml} \mathbb{A}^m$ . Second, change variables again

$$y_m \rightarrow \text{ch}\sigma_m = \frac{z^2(1 + e^{2ml}) + y_m^2}{2z^2 e^{ml}}. \tag{29}$$

The volume integral of a function of  $u \equiv \text{ch}\sigma_m$  then becomes

$$\begin{aligned} \int d\text{vol}_X \bullet &= \int_1^{e^l} \frac{dz}{z^{n+1}} \int_0^\infty dy_m y_m^{n-1} \text{vol}(S^{n-1}) \left\| \frac{\partial \vec{y}_m}{\partial \vec{x}} \right\|^{-1} \bullet \\ &= \int_1^{e^l} \frac{dz}{z^{n+1}} \int_0^\infty dy_m y_m^{n-1} \text{vol}(S^{n-1}) \left\| \frac{\partial \vec{y}_m}{\partial \vec{x}} \right\|^{-1} \bullet \\ &= 2^{n/2-1} e^{mln/2} \text{vol}(S^{n-1}) |\det(\mathbb{I} - e^{ml} \mathbb{A}^m)|^{-1} \\ &\quad \times \int_1^{e^l} \frac{dz}{z} \int_{\text{ch}ml}^\infty du (u - \text{ch}ml)^{n/2-1} \bullet. \end{aligned} \tag{30}$$

Here we note that the last expression is nothing but Weyl's fractional integral of order  $n/2$ , so that the volume integral of a function which only depends on the geodesic distance between image points can be cast into the following convenient form:

$$2^{n/2-1} l e^{mln/2} \text{vol}(S^{n-1}) |\det(\mathbb{I} - e^{ml} \mathbb{A}^m)|^{-1} \Gamma\left(\frac{n}{2}\right) \cdot {}_{\text{ch } ml} W_\infty^{-\frac{n}{2}}[\bullet]. \tag{31}$$

Inserting the heat kernel for the quotient space, obtained by summing over image locations, the composition of fractional integral and derivative easily gives the indirect contributions to the trace of the heat kernel:

$$\text{tr}' e^{-t\Delta_X} = l \sum_{m \in \mathbb{Z}, m \neq 0} e^{mln/2} |\det(\mathbb{I} - e^{ml} \mathbb{A}^m)|^{-1} \frac{1}{(4\pi t)^{1/2}} e^{-tn^2/4 - (ml)^2/4t}. \tag{32}$$

To get the trace of the Green's function, i.e. of the resolvent, it remains to take the proper-time integral. After straightforward manipulations and subtracting the analytically continued result from  $\lambda_+ = \lambda$  to  $\lambda_- = n - \lambda$ , we finally have

$$(2\lambda - n) \text{tr}' [G_X^+ - G_X^-] = -2l \sum_{m=1}^\infty \frac{e^{-\lambda ml} + e^{(\lambda-n)ml}}{|\det(\mathbb{I} - e^{ml} \mathbb{A}^m)|^{-1}}, \tag{33}$$

which is one of the many ways to express the log-derivative of the corresponding Selberg zeta function (cf appendix C):

$$(n - \lambda/2) \text{tr}' [G_X^+ - G_X^-] = \frac{Z'_\Gamma}{Z_\Gamma}(\lambda) + \frac{Z'_\Gamma}{Z_\Gamma}(n - \lambda). \tag{34}$$

In terms of the determinants, the result is compactly expressed as

$$\frac{\det'_-(\Delta_X - \lambda(n - \lambda))}{\det'_+(\Delta_X - \lambda(n - \lambda))} = [Z_\Gamma(n - \lambda)/Z_\Gamma(\lambda)]^2. \tag{35}$$



4.1. Renormalized volume and Euler characteristic

The prime on the determinants and traces above means that one has excluded the direct contribution, that is, the term containing the volume of the fundamental region in the bulk. This term requires renormalization, very much like in the exact hyperbolic case [17]. A renormalized version of the trace involves the renormalized volume  $\mathcal{V}$  and its anomaly  $\mathcal{L}$  under conformal rescaling with the very same coefficients as in the exact hyperbolic case [17] and reads

$$\text{tr} (G_H^+ - G_H^-) = \mathcal{A}_n \cdot \mathcal{V} + \mathcal{B}_n \cdot \mathcal{L}. \tag{36}$$

There are some general results on the geometric entries  $\mathcal{V}$  and  $\mathcal{L}$  that apply in the locally conformally flat case that is considered here (cf the appendix by Epstein in [33]). When  $n = \text{odd}$ ,  $\mathcal{L}$  is always absent. The renormalized volume vanishes (appendix D), consistent with the connection with the Euler characteristic  $\chi$  of the conformally compactified bulk manifold  $\bar{X}$ , which vanishes in this case because the manifold can be shrunk to a circle. When  $n = \text{even}$ , in general, the renormalized volume is not conformal invariant and has an anomaly  $\mathcal{L}$ , but in our case  $\mathcal{L}$  vanishes, consistent again with the vanishing Euler characteristic of  $\bar{X}$ , and the renormalized volume also vanishes in dimensional regularization with a minimal subtraction prescription (appendix D). There is, however, an important caveat concerning the arbitrariness of this subtraction: the ambiguity in this renormalized value  $\mathcal{V}$  is reflected in the holographic formula as an exponential term in front of the zeta factors

$$\exp \left\{ -\mathcal{V} \cdot \int_0^{\nu} d\tilde{\nu} 4\tilde{\nu} \mathcal{A}_n(\tilde{\nu}) \right\}. \tag{37}$$

In all, sticking to dimensional regularization with a minimal subtraction prescription, the direct contribution vanishes and we can drop the primes in all previous formulas.

5. BTZ and the Laplacian on the torus

As an illustration of the riches of the holographic formula, let us consider a classical result in the literature, namely, the determinant of the (scalar) Laplacian on the (two-) torus [34, 35], a crucial ingredient in the one-loop string amplitude resulting from the contributions of all closed surfaces with the topology of the torus to Polyakov’s path integral. The determinant, computed by Ray and Singer and later by Polchinski using standard  $\zeta$ -regularization, reads

$$\det' \Delta_g = e^{-\pi \tau_2/3} \tau_2^{-2} \prod_{k=1}^{\infty} |1 - e^{2\pi i k \tau}|^4, \tag{38}$$

where the prime means omission of the zero mode and  $\tau \equiv \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + i\frac{l}{2\pi}$  is the modular parameter of the torus. We will identify the holographic nature of each term in the above result. The exponential terms will be traced back to the renormalized volume of BTZ and the rest will result from the Selberg zeta factors.

5.1. GJMS and zero mode

First thing to note is the connection with GJMS operators (cf [36]). In the present case  $n = 2$  and when  $\nu \rightarrow 1$  (equivalently,  $\lambda \rightarrow 2$ ) the determinant of the scattering operator produces that of the first GJMS operator, i.e. the conformal Laplacian which in two dimensions is simply the Laplacian on functions. But the Laplacian has a zero mode coming from constant functions on the torus; therefore, one obtains a vanishing result in the limit unless the contribution from

this zero mode is excluded. It is easy to see from the elements of the scattering operators that the zero mode comes from the term  $L = \kappa = 0$  in (13) and (16):

$$\mathbb{S}_{0,0}(\lambda) = \frac{\Gamma^2(1/2 + \nu/2)}{\Gamma^2(1/2 - \nu/2)} \tag{39}$$

as  $\nu \rightarrow 1$ . The other matrix elements become the eigenvalues  $\kappa^2 + L^2$  of the Laplacian on the torus<sup>3</sup>. In higher dimensions, one obtains the eigenvalues of the *conformal Laplacian* instead.

### 5.2. Indirect contributions

The nontrivial determinant is therefore reached in the limit ( $\nu \equiv 1 - \epsilon$ )

$$\det' \Delta_g = \lim_{\epsilon \rightarrow 0} \det S_T(2 - \epsilon) \cdot \Gamma^2(\epsilon/2) = \lim_{\epsilon \rightarrow 0} \left[ \frac{Z_\Gamma(\epsilon)}{Z_\Gamma(2)} \right]^2 \cdot \Gamma^2(\epsilon/2). \tag{40}$$

Using the fact that  $Z_\Gamma(0) = 0$ , it is easy to recognize the derivative of the Selberg zeta function and a simple computation renders the following result:

$$\det' \Delta_g = 4 \left[ \frac{Z'_\Gamma(0)}{Z_\Gamma(2)} \right]^2 = (2l)^2 \prod_{k=1}^{\infty} |1 - e^{-kl+ik\theta}|^4. \tag{41}$$

Modulo an unimportant numerical actor  $(4\pi)^2$ , we get the second and third terms of the  $\zeta$ -regularized determinant. The first term is still unaccounted for, but we will immediately see its origin.

### 5.3. Direct contribution and renormalization

We have to recall the discussion in (4.1). The exponential term in the  $\zeta$ -regularized determinant comes from the term (37) with a choice of renormalized volume that coincides with the one obtained via the Hadamard or Riesz renormalization  $\mathcal{V} = -\pi l/2$  (D.6). The coefficient is

$$- \int_0^1 dv 4v \mathcal{A}_2(v) = \frac{1}{\pi} \int_0^1 dv v^2 = \frac{1}{3\pi}, \tag{42}$$

so that the direct contribution is responsible for the first term

$$e^{-l/6} = e^{-\pi\tau_2/3} \tag{43}$$

and the holographic description is completed.

## 6. Weierstrass regularization and quasi-normal frequencies

Recently, a related holographic recipe has been given to compute the Euclidean bulk determinant in terms of the spectrum of quasi-normal modes [37]. We will illustrate the connection to the holographic formula by examining the concrete example of BTZ<sub>3</sub>.

Let us recall the expression for the holographic formula that we get in this case:

$$\frac{\det_-(\Delta_{\text{BTZ}} - \lambda(2 - \lambda))}{\det_+(\Delta_{\text{BTZ}} - \lambda(2 - \lambda))} = \det S_T(\lambda) = \left[ \frac{Z_{\text{BTZ}}(2 - \lambda)}{Z_{\text{BTZ}}(\lambda)} \right]^2. \tag{44}$$

<sup>3</sup> In fact, these are the eigenvalues of the positive Laplacian modulo a factor 1/4 which is unimportant because the counting function vanishes in dimensional regularization (cf [17]).

The conformal boundary is a two-torus  $T$  with a parallelogram as a fundamental domain, described by a Teichmüller parameter, and  $Z_{\text{BTZ}}$  is the Selberg zeta function attached to the BTZ geometry by Perry and Williams [31],

$$Z_{\text{BTZ}}(\lambda) = \prod_{k_1, k_2 \geq 0} [1 - \alpha_1^{k_1} \alpha_2^{k_2} e^{-(k_1+k_2+\lambda)l}], \tag{45}$$

with  $\alpha_1 = e^{i\theta}$  and  $\alpha_2 = e^{-i\theta}$ . The relation to the standard parametrization of the spinning BTZ<sub>3</sub> [10] is given by

$$l = 2\pi r_+ \text{ and } \theta = 2\pi |r_-|. \tag{46}$$

It is direct to see that the set of zeros of  $Z_{\text{BTZ}}(\lambda)$  is given by

$$\mathcal{R} = \left\{ \zeta_{k_1, k_2, m} = -(k_1 + k_2) + i(k_1 - k_2)\theta + 2i\pi \frac{m}{l} \right\} \tag{47}$$

with  $k_{1,2} \in \mathbb{N}_0$  and  $m \in \mathbb{Z}$ . However, the nontrivial observation [31, 33] is that the set of the poles of the scattering operator (equation (13)),

$$\mathcal{R}' = \left\{ s_{m, N, j} = -2j - |m| \pm i \frac{2\pi N - m\theta}{l} \right\}, \tag{48}$$

with  $j \in \mathbb{N}_0$  and  $N, m \in \mathbb{Z}$ , exactly matches the set in equation (47). This can be referred to as *rudiments of holography*.

The Selberg zeta function  $Z_{\text{BTZ}}$  has therefore a Hadamard product representation [31], which can be thought of as a Weierstrass-regularized [38] version of the product of zeros,

$$Z_{\text{BTZ}}(\lambda) = e^{Q(\lambda)} \prod_{\zeta \in \mathcal{R}} (1 - \lambda/\zeta) e^{\lambda/\zeta + \frac{1}{2}(\lambda/\zeta)^2 + \frac{1}{3}(\lambda/\zeta)^3}, \tag{49}$$

where  $Q$  is a polynomial of degree at most three with finite coefficients. But due to the matching  $\mathcal{R} = \mathcal{R}'$ , one concludes then that the holographic formula produces a Weierstrass-regularized product of scattering resonances.

Now we come to the recipe in [37] and to their main observation that, for the non-spinning BTZ<sub>3</sub>, the set of scattering resonances can be rephrased as Matsubara plus quasi-normal frequencies. To see this, recall the spectrum of quasi-normal frequencies for a real scalar field in the non-spinning BTZ black hole:

$$\omega_{QN} = N - i l (\lambda + 2j) / 2\pi. \tag{50}$$

Note then that for the Euclidean version of the non-spinning BTZ<sub>3</sub>

$$\lambda - s_{m, N, j} = \lambda + 2j + 2\pi i N / l + |m| = |m| + 2\pi i \omega_{QN} / l, \tag{51}$$

so that the naive product of resonances can be rewritten in terms of the quasi-normal frequencies as follows:

$$\prod_{\zeta \in \mathcal{R}} (\lambda - \zeta) = \prod_{s \in \mathcal{R}'} (\lambda - s) = \prod_{\text{Matsubara, } QN} (|m| + 2\pi i \omega_{QN} / l). \tag{52}$$

The partial product over  $m \in \mathbb{Z}$  (Matsubara frequencies) can be regularized using the gamma function, and the product can be cast as a product over quasi-normal frequencies of gamma functions; however, the resulting expression is ill-defined because the infinite product is still divergent.

The proposal in [37] is for the Euclidean determinant in the bulk, whereas the holographic formula gives the answer with respect to a reference (the continuation to  $n - \lambda$ ). The computation of the separate determinants in the bulk requires the regularization/renormalization of the UV divergence in the bulk, which was bypassed by the

holographic formula that matches IR in the bulk to UV on the boundary. Now, some caution must be taken in trying to separate the pieces of the holographic formula. The situation is in fact very reminiscent to the computation of the scalar field exchange Witten graph in the early days of AdS/CFT correspondence. The duality naturally gives an expression for the difference of exchange graphs involving bulk-to-bulk propagators for  $\lambda$  and  $n - \lambda$  which matches exactly the difference of the conformal partial waves of the dual operator with scaling dimension  $\lambda$  and its conjugate with  $n - \lambda$ . Initially, assuming analyticity, each exchange amplitude and the corresponding conformal partial wave were claimed to be identical [39]; however, the presence of logarithmic terms that spoiled the identification was later realized [40].

An expression in terms of quasi-normal frequencies is appealing because it can be extended beyond the exact cases considered here, but the scattering resonances seem to play the central role. In the spinning BTZ<sub>3</sub>, where the temperature circle gets mixed with other directions, the connection between quasi-normal frequencies and scattering resonances becomes obscured, and we believe that the last ones are more directly connected with the Euclidean determinant.

A final remark concerning the exact cases is considered here. The Weierstrass regularization is a prescription to render the products of resonances finite and produces the Selberg zeta function. This can be achieved invoking Barnes' zeta function (cf [41]) and the determinant, roughly  $\exp\{-\zeta'(0)\}$ , is naturally expressed in terms of Barnes' multiple gamma function (cf [41]). The holographic formula then suggests a nontrivial relation between the (Patterson–)Selberg zeta functions and Barnes' multiple gammas, which in fact has been established by other routes (equation (6.4) in [41])<sup>4</sup>.

## 7. Conclusion

In this paper we have verified the holographic formula for bulk geometries obtained by identifications in the hyperbolic space which combine dilation with internal rotations. The quotient geometries include the Euclidean section of the higher dimensional generalization of the non-spinning BTZ black hole and the functional determinants are expressed in term of the associated Selberg zeta function. A connection with quasi-normal frequencies has been elucidated and it seems very likely that these functional determinants can be written in terms of the Selberg zeta function for all cases where the spectrum of quasi-normal frequencies is explicitly known. The existence of quasi-normal modes for spinor, vector and tensor excitations in the bulk strongly suggests that there must be suitable versions of the holographic formula relating one-loop bulk determinants to those of the corresponding boundary two-point functions.

The exact expressions in terms of the Selberg zeta function, more than just a mathematical curiosity, are suitable to analytically explore the low- and high-temperature regimes, on the one hand. On the other hand, much of the work done using WKB approximation in cases that deviate from the exact ones considered here, should correspond to the transition from the Selberg zeta function and trace formula to a semiclassical zeta function and Gutzwiller trace formula (cf [42]).

Finally, let us mention a newly found gauge–gravity relation between the one-loop effective action for a charged scalar in a maximally symmetric background electromagnetic field and the one-loop effective action for a spinor in Euclidean AdS [43]. It is known that on the gauge side, Barnes' zeta and gamma functions arise in less symmetric situations. Now, we have argued that there must be a corresponding holographic formula for spinors and the Selberg zeta function should play a central role; therefore, the connection with Barnes'

<sup>4</sup> We are indebted to E Friedman for clarification of this point. Further elaboration will be presented somewhere else.

functions strongly suggests that the backgrounds considered in this paper are natural candidates for the gravitational counterpart of these less symmetric electromagnetic backgrounds or to their finite-temperature versions.

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### Appendix A. Constant curvature spaces

In general a constant curvature space can be written in terms of

$$ds^2 = B(\rho)^2(\hat{e}^i \hat{e}^j \eta_{ij}) + C(\rho)^2 d\rho^2 + D(\rho)^2(\tilde{e}^m \tilde{e}^n \eta_{mn}), \quad (\text{A.1})$$

or in terms of the vielbein

$$\begin{aligned} e^i &= B(\rho)\hat{e}^i \\ e^r &= C(\rho) d\rho \\ e^n &= D(\rho)\tilde{e}^n. \end{aligned}$$

Since this is a torsion-free solution,  $T^a = 0$ ; therefore, the connection is given by

$$\begin{aligned} w^{ij} &= \hat{w}^{ij} \\ w^{ir} &= \frac{B(\rho)'}{C(\rho)B(\rho)} e^i \\ w^{mr} &= \frac{D(\rho)'}{C(\rho)D(\rho)} e^m \\ w^{mn} &= \tilde{w}^{mn}, \end{aligned}$$

and so the curvatures are completely determined by the intrinsic curvatures as

$$\begin{aligned} R^{ij} &= \hat{R}^{ij} - \left( \frac{\ln(B(\rho))'}{C(\rho)} \right)^2 e^i \wedge e^j \\ R^{ir} &= -\frac{1}{B(r)C(r)} \left( \frac{B(\rho)'}{C(\rho)} \right)' e^i \wedge e^r \\ R^{im} &= -\frac{\ln(B(r))' \ln(D(\rho))'}{C(\rho)^2} e^i \wedge e^m \\ R^{rm} &= -\frac{1}{A(r)C(r)} \left( \frac{D(\rho)'}{C(\rho)} \right)' e^r \wedge e^m \\ R^{mn} &= \tilde{R}^{mn} - \left( \frac{\ln(D(\rho))'}{C(\rho)} \right)^2 e^m \wedge e^n. \end{aligned}$$

This solution has still the invariance of the definition of the  $\rho$  coordinate. By fixing  $C(\rho) = l(2\rho)^{-1}$ , as in the usual Fefferman–Graham coordinates, the constant curvature solution, *i.e.*  $\tilde{R}^{ab} = R^{ab} + l^{-2}e^a e^b = 0$ , is given by

$$\begin{aligned} B(\rho) &= \frac{l\sqrt{2}}{4\sqrt{\rho\beta}(c_1 - c_2)} (\rho(c_1 - c_2) + 2) \\ D(\rho) &= \frac{l\sqrt{2}}{4\sqrt{\rho\alpha}(c_2 - c_1)} (\rho(c_2 - c_1) + 2) \end{aligned}$$

where the intrinsic curvatures are given by

$$\hat{R}^{ij} = -\frac{1}{\beta} \hat{e}^i \hat{e}^j \quad \text{and} \quad \tilde{R}^{mn} = -\frac{1}{\alpha} \tilde{e}^m \tilde{e}^n$$

with  $\alpha = -\beta = \pm 1$  and  $c_1$  and  $c_2$  are constant to be determined.

### Appendix B. From the quadratic line element to Poincaré coordinates

In order to write explicitly the identifications defined by equation (4), it is useful to write the quadratic form

$$-p^2 + w^2 + (u^1)^2 + \dots + (u^{d-1})^2 = -1. \tag{B.1}$$

The Poincaré coordinate set for  $H_d$  arises from

$$\begin{aligned} p &= \frac{1}{2z} (z^2 + (y^1)^2 + \dots + (y^d) + 1) \\ w &= \frac{1}{2z} (z^2 + (y^1)^2 + \dots + (y^d) - 1) \\ y^i &= \frac{u^i}{2z} \end{aligned}$$

with  $i = 1, \dots, d - 1$ . The Poincaré half-plane is therefore described by

$$ds^2 = \frac{1}{z^2} (dz^2 + \delta_{ij} dy^i dy^j) \tag{B.2}$$

reproducing equation (3).

In this coordinates the boost in the plane  $w, p$ , generated by the Killing vector

$$\xi = w \frac{\partial}{\partial p} + p \frac{\partial}{\partial w}, \tag{B.3}$$

is given by

$$\xi = z \frac{\partial}{\partial z} + y^i \frac{\partial}{\partial y^i}.$$

Obviously the generator  $g = e^{l\xi}$  represents a dilatation in equation (B.2). On the other hand, one can note that the set of all rotation that commute among themselves and with  $\xi$  is given, up to some global rotation, by rotations in each of the planes  $(u^{k_1}, u^{k_2}) = \{(u^1, u^2), (u^3, u^4), \dots, (u^{j-1}, u^j)\}$ , with  $j$  the integer part of  $(d - 1)/2$ . These rotations are generated by

$$\zeta_k = \left( u^{k_1} \frac{\partial}{\partial u^{k_2}} - u^{k_2} \frac{\partial}{\partial u^{k_1}} \right), \tag{B.4}$$

which in the Poincaré coordinate can be written as

$$\zeta_k = \left( y^{k_1} \frac{\partial}{\partial y^{k_2}} - y^{k_2} \frac{\partial}{\partial y^{k_1}} \right).$$

### Appendix C. Selberg zeta function

The Selberg zeta function associated with the quotient geometries we consider was first introduced by Patterson [19] in the form of the Euler product. In terms of the length  $l$  of the (primitive) closed geodesic and the eigenvalues  $\{\alpha_1, \dots, \alpha_n\}$  of the rotation matrix  $\mathbb{A}$ , it is given by

$$Z_\Gamma(\lambda) = \prod_{k_1, \dots, k_n \geq 0} [1 - \alpha_1^{k_1} \dots \alpha_n^{k_n} e^{-(k_1 + \dots + k_n + \lambda)l}]. \quad (C.1)$$

This Selberg zeta function has also an interpretation as the dynamical zeta function for geodesic flow on  $X = \Gamma \backslash \mathbb{H}^{n+1}$  [31]. The matrix  $\mathbb{A}$  describes the rotation of nearby closed geodesics under the Poincaré once-return map  $\mathcal{P}_\gamma$ . Elementary manipulations lead to

$$\log Z_\Gamma(\lambda) = - \sum_{m \geq 1} \frac{1}{m} \frac{e^{-m\lambda}}{\det(\mathbb{I} - e^{-m\lambda} \mathbb{A}^m)}, \quad (C.2)$$

with

$$\det(\mathbb{I} - e^{-m\lambda} \mathbb{A}^m) = \prod_{j=1}^n (1 - \alpha_j^m e^{-m\lambda}) = e^{-m\lambda n/2} |\det(\mathbb{I} - \mathcal{P}_\gamma^m)|^{1/2}. \quad (C.3)$$

### Appendix D. Renormalized volume of BTZ

To renormalize the volume of the quotient space  $X$ , it is convenient first to cast the metric into the Fefferman–Graham form with  $r = \frac{1-s^2/4}{s} = \text{sh} \ln \frac{2}{s}$ ,

$$g_X = s^{-2} \{ ds^2 + (1 + s^2/4)^2 du^2 + (1 - s^2/4)^2 d\Omega_{n-1}^2 \}. \quad (D.1)$$

The computation in dimensional regularization gives an exactly vanishing answer

$$\int d\text{vol}_X = \text{vol}(\mathcal{M}) \cdot \int_0^2 ds s^{-n-1} (1 - s^2/4)^{n-1} (1 + s^2/4) = 0. \quad (D.2)$$

The cutoff or Hadamard regularization in the case  $n = \text{even}$  produces a different renormalized volume. We present here the Riesz regularization [44] which produces, in fewer steps, the same answer as Hadamard regularization. The trick is to insert  $s^z$  to regularize the divergence at  $s = 0$  and analytically continue in  $z$  to obtain the result at  $z = 0$

$$\begin{aligned} \int d\text{vol}_X &= \text{vol}(\mathcal{M}) \cdot \int_0^2 ds s^{z-n-1} (1 - s^2/4)^{n-1} (1 + s^2/4) \\ &= \text{vol}(\mathcal{M}) \cdot \frac{1}{2^{n+1-z}} \Gamma(n) \frac{z \Gamma(z/2 - n/2)}{\Gamma(z/2 + n/2 + 1)}. \end{aligned} \quad (D.3)$$

When  $n = \text{odd}$  this clearly vanishes, consistent with the fact that the Euler characteristic of  $\mathcal{M}$  also vanishes because of the  $S^1$  factor. When  $n = \text{even}$ , the asymptotic behavior as  $z \rightarrow 0$  is

$$\int d\text{vol}_X = (-1)^{n/2} \text{vol}(\mathcal{M}) \cdot \frac{\Gamma(n)}{2^n} + O(z), \quad (D.4)$$

and one can read off a vanishing anomaly due to the absence of a log-term, consistent with the vanishing Euler characteristic of the conformal boundary, and the renormalized volume

$$\mathcal{V}_X = (-1)^{n/2} \text{vol}(\mathcal{M}) \cdot \frac{\Gamma(n)}{2^n}. \quad (D.5)$$

In the specific case of the spinning BTZ<sub>3</sub>, the renormalized volume using this scheme is

$$\mathcal{V}_{\text{BTZ}} = -\pi l/2 = -\pi^2 r_+ = -\pi^2 \tau_2. \quad (D.6)$$

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